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Note

A generalised beta integral and the limit of the Bernstein–Durrmeyer operator with Jacobi weights

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Abstract

We give a generalisation of the multivariate beta integral. This is used to show that the (multivariate) Bernstein–Durrmeyer operator for a Jacobi weight has a limit as the weight becomes singular. The limit is an operator previously studied by Goodman and Sharma. From the elementary proof given, it follows that this operator inherits many properties of the Bernstein–Durrmeyer operator in a natural way. In particular, we determine its eigenstructure and give a differentiation formula for it which is new.

Keywords: Bernstein–Durrmeyer operator; Multivariate beta integral; Jacobi polynomials

1. Introduction

The Bernstein operator $B_n : C[0,1] \to \Pi_n$, which is defined by

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \tag{1.1}$$

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can be modified to obtain M_n^μ the Bernstein–Durrmeyer operator for a Jacobi weight

$$M_n^\mu f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} c_k^\mu(f),$$

where

$$c_k^\mu(f) := \frac{\int_0^1 x^{k+\mu_1}(1-x)^{n-k+\mu_2} f(x)\,\mathrm{d}x}{\int_0^1 x^{k+\mu_1}(1-x)^{n-k+\mu_2}\,\mathrm{d}x}, \quad \mu = (\mu_1, \mu_2), \ \ \mu_1, \mu_2 > -1.$$

The multivariate version of this operator, where the interval $[0,1]$ is replaced by a simplex T in $\mathbb{R}^s$, is defined below. By using a generalisation of the multivariate beta integral

$$\int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{s-1}} x_1^{\beta_1-1} \cdots x_s^{\beta_s-1} (1 - x_1 - \cdots - x_s)^{\beta_0-1}\,\mathrm{d}x_s \ldots \mathrm{d}x_2\,\mathrm{d}x_1$$
$$= \frac{\Gamma(\beta_0)\Gamma(\beta_1)\cdots\Gamma(\beta_s)}{\Gamma(\beta_0+\beta_1+\cdots+\beta_s)}, \quad \beta_0, \beta_1, \ldots, \beta_s > 0, \tag{1.2}$$

we show $c_k^\mu(f)$ and its multivariate analogue have a limit as some or all $\mu_i \to -1^+$. Thus, $M_n^\mu f$ converges as $\mu \to -1^+$, with the limit operator inheriting many properties in a natural way. It turns out that this limit operator is the one previously studied by Chen in the univariate case, and Goodman–Sharma in the multivariate case. Hence, we provide a simple explanation as to why this operator has properties which are so close to those of the Bernstein–Durrmeyer operator, and a simple proof of these properties. In particular, we determine its eigenstructure and give a differentiation formula for it which is new.

1.2. Definitions

Let V be the set of $s+1$ vertices of an s-simplex T in $\mathbb{R}^s$, and $\xi = (\xi_v)_{v \in V}$ be the corresponding barycentric coordinates. We will use standard multi-index notation with indices from $\mathbb{Z}_+^V$ and $\mathbb{Z}_+^s$, so, for example,

$$\xi^\alpha := \prod_{v \in V} \xi_v^{\alpha_v}, \quad \alpha \in \mathbb{Z}_+^V, \qquad \beta! := \beta_1!\beta_2!\cdots\beta_s!, \quad \beta \in \mathbb{Z}_+^s.$$

The value of α at v is denoted by α_v or $\alpha(v)$, depending on which is most aesthetic.

The *Bernstein operator* of degree n for the simplex T with vertices V is defined by

$$B_{n,V} f := \sum_{\substack{|\alpha|=n \\ \alpha \in \mathbb{Z}_+^V}} \binom{n}{\alpha} \xi^\alpha f(v_\alpha), \quad \forall f \in C(T), \tag{1.3}$$

where

$$\binom{n}{\alpha} := \frac{n!}{\alpha!(n-|\alpha|)!}, \quad v_\alpha := \sum_{v \in V} \frac{\alpha(v)}{|\alpha|} v \in T.$$

To describe Durrmeyer's modification of this operator, we define the linear functional

$$f \mapsto \int_{[\theta_0,\dots,\theta_k]} f := \frac{1}{k!\,\mathrm{vol}_k(S)} \int_S f \circ A, \tag{1.4}$$

where S is any k-simplex in $\mathbb{R}^d$ with (k-dimensional) volume $\mathrm{vol}_k(S)$, and $A : \mathbb{R}^d \to \mathbb{R}^s$ is any affine map taking the $k+1$ vertices of S onto the points $\theta_0, \dots, \theta_k$ in $\mathbb{R}^s$ (this is independent of the choice of S and A). For the points V, taking $S = T$ and A as the identity gives

$$\int_V f = \frac{1}{s!\,\mathrm{vol}_s(T)} \int_T f. \tag{1.5}$$

For $\mu_v > -1$, $\forall v \in V$, the weight $\xi^\mu := \prod_{v\in V} \xi_v^{\mu_v}$ is integrable on T. We denote the corresponding weighted inner product space by $L_2(T, \xi^\mu)$, and use the inner product

$$\langle f, g\rangle_\mu := \int_V fg\xi^\mu = \frac{1}{s!\,\mathrm{vol}_s(T)} \int_T fg\xi^\mu, \quad \forall f, g \in L_2(T, \xi^\mu).$$

The *Bernstein–Durrmeyer operator* of degree n for a Jacobi weight ξ^μ on T can now be defined by

$$M_n^\mu f := \sum_{|\alpha|=n} \binom{n}{\alpha} \xi^\alpha \frac{\langle f, \xi^\alpha\rangle_\mu}{\langle 1, \xi^\alpha\rangle_\mu}, \quad \forall f \in L_2(T, \xi^\mu). \tag{1.6}$$

This self-adjoint operator on $L_2(T, \xi^\mu)$ was first defined on $L_2[0,1]$ by Durrmeyer [Du67] (see also [D81]), then for Jacobi weights by Păltănea [P83] (see also [BX91]), and the multivariate analogues in Derriennic [D85] (see also [Di95]).

2. A generalised multivariate beta integral

The multivariate gamma and beta functions of $\beta \in \mathbb{Z}_+^V$ are defined by

$$\Gamma(\beta) := \prod_{v\in V} \Gamma(\beta_v), \qquad B(\beta) := \frac{\Gamma(\beta)}{\Gamma(|\beta|)}, \quad \beta > 0.$$

The beta integral (1.2) can be written as

$$\int_V \xi^{\beta-1} = B(\beta) = \frac{\Gamma(\beta)}{\Gamma(|\beta|)}, \quad \beta > 0. \tag{2.1}$$

If $\theta_0, \dots, \theta_k$ of (1.4) are the points $V = \{v_0, \dots, v_s\}$ taken with multiplicities $\alpha(v_i) \geqslant 0$, $\alpha \neq 0$, then a change of variables shows that

$$\int_{[\underbrace{v_0,\dots,v_0}_{\alpha(v_0)},\dots,\underbrace{v_s,\dots,v_s}_{\alpha(v_s)}]} f = \frac{1}{\Gamma(\alpha_|)} \int_{\mathrm{supp}(\alpha)} f\,\xi_|^{\alpha_|-1}, \qquad \begin{aligned} \xi_| &:= (\xi_v)_{v\in\mathrm{supp}(\alpha)}, \\ \alpha_| &:= \alpha|_{\mathrm{supp}(\alpha)}, \end{aligned} \tag{2.2}$$

where $\mathrm{supp}(\alpha) \subset V$ denotes the support of α.

Proposition 2.1 (Generalised beta integral). *Let* $\alpha \in \mathbb{Z}_+^V$, $\kappa \in \mathbb{R}^V$. *For* $\kappa > -\alpha$, $\alpha \geqslant 1$

$$\int_{[\underbrace{v_0,\,\dots,\,v_0}_{\alpha(v_0)},\dots,\underbrace{v_s,\,\dots,\,v_s}_{\alpha(v_s)}]} \xi^\kappa = \frac{B(\alpha+\kappa)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+\kappa)}{\Gamma(|\alpha+\kappa|)\Gamma(\alpha)}. \tag{2.3}$$

Proof. Take $f = \xi^\kappa$ in (2.2) and use the beta integral (2.1), to obtain

$$\int_{[\underbrace{v_0,\,\dots,\,v_0}_{\alpha(v_0)},\dots,\underbrace{v_s,\,\dots,\,v_s}_{\alpha(v_s)}]} \xi^\kappa = \frac{1}{\Gamma(\alpha)} \int_V \xi^{\alpha+\kappa-1} = \frac{B(\alpha+\kappa)}{\Gamma(\alpha)}. \qquad \square$$

For $\alpha = 1$ and $\kappa = \beta - 1$, the integral (2.3) reduces to the classical beta integral (2.1).

3. The limit of the Bernstein–Durrmeyer operator

The inner product $\langle 1, \xi^\alpha \rangle_\mu$, $\mu > -1$ in (1.6) becomes unbounded as any component of μ approaches -1, and so a limiting form of $M_n^\mu f$ as $\mu \to -1^+$ ($\mu_v \to -1^+$, $\forall v \in V$) cannot be defined by substituting $\mu = -1$ into (1.6). However, for $f \in C(T)$, we show that

$$\lim_{\mu \to -1^+} \frac{\langle f, \xi^\alpha \rangle_\mu}{\langle 1, \xi^\alpha \rangle_\mu} = (n-1)! \int_{[\underbrace{v_0,\,\dots,\,v_0}_{\alpha(v_0)},\dots,\underbrace{v_s,\,\dots,\,v_s}_{\alpha(v_s)}]} f,$$

and so a limiting form can be defined in a natural way.

The multivariate shifted factorial (Pochhammer symbol) is defined by

$$(\mu)_\alpha := \prod_{v \in V} (\mu_v)_{\alpha_v}, \quad \alpha \in \mathbb{Z}_+^V, \qquad (\mu_v)_{\alpha_v} := \mu_v(\mu_v+1)\cdots(\mu_v+\alpha_v-1),$$

and satisfies

$$\frac{\Gamma(\mu+\alpha)}{\Gamma(\mu)} = (\mu)_\alpha, \quad \mu > 0. \tag{3.1}$$

We extend $|\mu|$ to vectors $\mu \in \mathbb{R}^V$ (which may have negative entries) via $|\mu| := \sum_{v \in V} \mu_v$.

Lemma 3.1. *Let* $\kappa \in \mathbb{R}^V$, *with* $\kappa \geqslant -1$, *and define* $\lceil -\kappa \rceil \in \mathbb{Z}_+^V$ *by*

$$\lceil -\kappa \rceil(v) := \begin{cases} 1, & \kappa_v = -1, \\ 0, & \kappa_v > -1. \end{cases}$$

Then, for $f \in C(T)$ and $\alpha \in \mathbb{Z}_+^V$, $|\alpha| = n \geqslant 1$, we have

$$\lim_{\substack{\mu \to \kappa \\ \mu > -1}} \frac{\langle f, \xi^\alpha \rangle_\mu}{\langle 1, \xi^\alpha \rangle_\mu} = c_\alpha^\kappa(f) := \frac{1}{C_{\alpha,\kappa}} \int_{[\underbrace{v_0, \dots, v_0}_{\alpha(v_0)+1-\lceil -\kappa \rceil(v_0)}, \dots, \underbrace{v_s, \dots, v_s}_{\alpha(v_s)+1-\lceil -\kappa \rceil(v_s)}]} f \xi^{\kappa + \lceil -\kappa \rceil}, \tag{3.2}$$

where

$$C_{\alpha,\kappa} := \int_{[\underbrace{v_0, \dots, v_0}_{\alpha(v_0)+1-\lceil -\kappa \rceil(v_0)}, \dots, \underbrace{v_s, \dots, v_s}_{\alpha(v_s)+1-\lceil -\kappa \rceil(v_s)}]} \xi^{\kappa + \lceil -\kappa \rceil} = \frac{\Gamma(\alpha + \kappa + 1)}{\Gamma(|\alpha| + |\kappa| + s + 1)\Gamma(\alpha + 1 - \lceil -\kappa \rceil)}.$$

In particular,

$$\lim_{\mu \to -1^+} \frac{\langle f, \xi^\alpha \rangle_\mu}{\langle 1, \xi^\alpha \rangle_\mu} = (n-1)! \int_{[\underbrace{v_0, \dots, v_0}_{\alpha(v_0)}, \dots, \underbrace{v_s, \dots, v_s}_{\alpha(v_s)}]} f. \tag{3.3}$$

Proof. The integrals defining $c_\alpha^\kappa(f)$ in (3.2) are finite since $\kappa + \lceil -\kappa \rceil > -1$. It suffices to prove (3.2) for the polynomials $f = \xi^\beta$, $\beta \in \mathbb{Z}_+^V$, since their span is dense in $C(T)$ and

$$\left| \frac{\langle f, \xi^\alpha \rangle_\mu}{\langle 1, \xi^\alpha \rangle_\mu} \right| \leqslant \|f\|_{\infty,T}, \qquad |c_\alpha^\kappa(f)| \leqslant \|f\|_{\infty,T}.$$

For the left-hand side, use the beta integral (2.1) and (3.1), to obtain

$$\lim_{\substack{\mu \to \kappa \\ \mu > -1}} \frac{\langle \xi^\beta, \xi^\alpha \rangle_\mu}{\langle 1, \xi^\alpha \rangle_\mu} = \lim_{\substack{\mu \to \kappa \\ \mu > -1}} \frac{(\alpha + \mu + 1)_\beta}{(|\alpha| + |\mu| + s + 1)_{|\beta|}} = \frac{(\alpha + \kappa + 1)_\beta}{(|\alpha| + |\kappa| + s + 1)_{|\beta|}}. \tag{3.4}$$

Note, for each $v \in V$, we have

$$(\alpha + 1 - \lceil -\kappa \rceil)_v = 0 \iff \alpha_v = 0,\ \kappa_v = -1 \iff (\alpha + \kappa + 1)_v = 0, \tag{3.5}$$

giving

$$W := \operatorname{supp}(\alpha + 1 - \lceil -\kappa \rceil) = \operatorname{supp}(\alpha + \kappa + 1), \tag{3.6}$$

and

$$(\kappa + \lceil -\kappa \rceil)|_{V \setminus W} = 0, \tag{3.7}$$

$$(\alpha + \kappa + 1)|_{V \setminus W} = 0. \tag{3.8}$$

Case 1: $W = V$. Then (3.6) implies

$$\alpha + 1 - \lceil -\kappa \rceil \geqslant 1, \quad \alpha + \kappa + 1 > 0$$

and so the generalised beta integral (2.3) gives

$$\int_{[\underbrace{v_0,\ldots,v_0}_{\alpha(v_0)+1-\lceil -\kappa\rceil(v_0)},\ldots,\underbrace{v_s,\ldots,v_s}_{\alpha(v_s)+1-\lceil -\kappa\rceil(v_s)}]} \xi^\beta \xi^{\kappa+\lceil -\kappa\rceil} = \frac{\Gamma(\alpha+1-\lceil -\kappa\rceil+\beta+\kappa+\lceil -\kappa\rceil)}{\Gamma(|\alpha|+|\beta|+|\kappa|+s+1)\Gamma(\alpha+1-\lceil -\kappa\rceil)}.$$

From this we calculate

$$c_\alpha^\kappa(f)=\frac{\Gamma(\alpha+\beta+\kappa+1)}{\Gamma(\alpha+\kappa+1)}\frac{\Gamma(|\alpha|+|\kappa|+s+1)}{\Gamma(|\alpha|+|\beta|+|\kappa|+s+1)}=\frac{(\alpha+\kappa+1)_\beta}{(|\alpha|+|\kappa|+s+1)_{|\beta|}}.$$

Case 2: $W\neq V$. Let $\xi_U^\alpha := (\xi_u)_{u\in U}^{\alpha|_U}=\prod_{u\in U}\xi_u^{\alpha_u}$ for $U\subset V$. By (2.2) and (3.7), we have

$$\begin{aligned}\int_{[\underbrace{v_0,\ldots,v_0}_{\alpha(v_0)+1-\lceil -\kappa\rceil(v_0)},\ldots,\underbrace{v_s,\ldots,v_s}_{\alpha(v_s)+1-\lceil -\kappa\rceil(v_s)}]} \xi^\beta \xi^{\kappa+\lceil -\kappa\rceil} &= \frac{1}{\Gamma((\alpha+1-\lceil -\kappa\rceil)|_W)}\int_W \xi^{\beta+\kappa+\lceil -\kappa\rceil}\xi_W^{\alpha-\lceil -\kappa\rceil}\\ &= \frac{1}{\Gamma((\alpha+1-\lceil -\kappa\rceil)|_W)}\int_W \xi_{V\backslash W}^\beta \xi_W^{\alpha+\beta+\kappa}.\end{aligned}$$

Suppose $\beta_v>0$ for some $v\in V\backslash W$. Then $\xi_{V\backslash W}^\beta$ has ξ_v as a factor, and hence is zero over the region of integration (the convex hull of W), giving

$$c_\alpha^\kappa(f)=0=\frac{(\alpha+\kappa+1)_\beta}{(|\alpha|+|\kappa|+s+1)_{|\beta|}},$$

with the last equality following since $(\alpha+\kappa+1)_v=0$ by (3.8).

Suppose $\beta_v=0$ for all $v\in V\backslash W$, i.e., $\mathrm{supp}(\beta)\subset W$ and $|\beta_{|_W}|=|\beta|$. Then $\xi_{V\backslash W}^\beta=1$, and we use the beta integral (2.1) to calculate

$$c_\alpha^\kappa(f)=\frac{\int_W \xi^{\alpha+\beta+\kappa}}{\int_W \xi^{\alpha+\kappa}}=\frac{\Gamma((\alpha+\beta+\kappa+1)_{|_W})}{\Gamma((\alpha+\kappa+1)_{|_W})}\frac{\Gamma(|(\alpha+\kappa+1)_{|_W}|)}{\Gamma(|(\alpha+\beta+\kappa+1)_{|_W}|)}.$$

The first factor in the above product is $(\alpha+\kappa+1)_\beta$ since $\mathrm{supp}(\beta)\subset W$. From (3.8), it follows that $|(\alpha+\kappa+1)_{|_{V\backslash W}}|=0$, and so, since $|\beta_{|_W}|=|\beta|$, the second factor becomes

$$\begin{aligned}\frac{\Gamma(|(\alpha+\kappa+1)_{|_W}|+|(\alpha+\kappa+1)_{|_{V\backslash W}}|)}{\Gamma(|(\alpha+\kappa+1)_{|_W}|+|\beta_{|_W}|+|(\alpha+\kappa+1)_{|_{V\backslash W}}|)} &= \frac{\Gamma(|\alpha|+|\kappa|+s+1)}{\Gamma(|\alpha|+|\beta|+|\kappa|+s+1)}\\ &= \frac{1}{(|\alpha|+|\kappa|+s+1)_{|\beta|}},\end{aligned}$$

as required.

Finally, the particular case (3.3) is obtained by taking $\kappa=-1$ in (3.2). $\square$

By (2.2), we have

$$c_\alpha^\kappa(f) = \frac{\langle f, \xi^\alpha \rangle_\kappa}{\langle 1, \xi^\alpha \rangle_\kappa}, \quad \kappa > -1, \tag{3.9}$$

so that $\{\kappa \in \mathbb{R}^V : \kappa \geqslant -1\} \to \mathbb{R} : \kappa \mapsto c_\alpha^\kappa(f)$ is continuous, and we have the following extension of M_n^μ to ‘singular weights’ $\xi^\kappa, \ \kappa \geqslant -1$.

Theorem 3.2 (Limit operator). *For $f \in C(T)$ and $\kappa \geqslant -1$, we have*

$$\lim_{\substack{\mu \to \kappa \\ \mu > -1}} M_n^\mu f = \hat{M}_n^\kappa f := \sum_{|\alpha|=n} \binom{n}{\alpha} \xi^\alpha c_\alpha^\kappa(f), \tag{3.10}$$

where $c_\alpha^\kappa(f)$ is given by (3.3), *and*

$$\hat{M}_n^\kappa = M_n^\kappa|_{C(T)}, \quad \kappa > -1. \tag{3.11}$$

In particular,

$$\lim_{\mu \to -1^+} M_n^\mu f = U_n f := (n-1)! \sum_{|\alpha|=n} \binom{n}{\alpha} \xi^\alpha \int_{[\underbrace{v_0, \ldots, v_0}_{\alpha(v_0)}, \ldots, \underbrace{v_s, \ldots, v_s}_{\alpha(v_s)}]} f. \tag{3.12}$$

Proof. For $\mu > -1$, Lemma 3.1 gives

$$||M_n^\mu f - \hat{M}_n^\kappa f||_{\infty,T} \leqslant \sum_{|\alpha|=n} \binom{n}{\alpha} ||\xi^\alpha||_{\infty,T} \left| \frac{\langle f, \xi^\alpha \rangle_\mu}{\langle 1, \xi^\alpha \rangle_\mu} - c_\alpha^\kappa(f) \right| \to 0, \quad \mu \to \kappa,$$

and (3.9) gives (3.11). Take $\kappa = -1$ in (3.10) to get (3.12). □

The operator $U_n : C(T) \to \Pi_n$ defined by (3.12) is due to Goodman and Sharma [GS91] (for the univariate case see [GS87]). It was also considered by Sauer [S94] who remarks the univariate version was known to W.Z. Chen in 1987. Since U_n is the limit of Durrmeyer operators $M_n^\mu, \ \mu > -1$, many properties of the Durrmeyer operators are inherited, and these can be proved (simply) by taking the limit as $\mu \to -1^+$, e.g.,

(i) The M_n^μ are positive linear operators on $C(T)$ with $||M_n^\mu f||_{\infty,T} \leqslant ||f||_{\infty,T}$.
(ii) They are degree reducing, i.e., $M_n^\mu(\Pi_k) \subset \Pi_k, \ \forall n, k$.
(iii) They commute, i.e., $M_n^\mu M_k^\mu = M_k^\mu M_n^\mu, \ \forall n, k$.

hold with U_n (or $\hat{M}_n^\kappa, \ \kappa \geqslant -1$) replacing M_n^μ. We illustrate this method in the next two sections by determining the eigenstructure of U_n and a differentiation formula for it.

Recently, Theorem 3.2 was obtained independently in the univariate case $s = 1$ ($\mu = (a, a) \to -1^+$) by Păltănea [P01], where the limit operator U_n was attributed to yet another: Gavrea [G96]. There it was shown that M_n^μ does not converge to $\hat{M}_n^\kappa$ in the operator norm. The example used can be modified to show that the multivariate M_n^μ does not converge to $\hat{M}_n^\kappa$ in the operator norm as soon as some $\kappa_v = -1$.

4. The eigenstructure of U_n

We now describe the eigenvalues and eigenspaces of U_n by taking the limit of those for M_n^μ. The eigenvalues of M_n^μ, $\mu > -1$ are

$$\lambda_k(M_n^\mu) := \frac{n!}{(n-k)!} \frac{\Gamma(n + |\mu| + s + 1)}{\Gamma(n + k + |\mu| + s + 1)}, \quad k = 0, 1, \ldots, n,$$

and the corresponding eigenfunctions are the Jacobi polynomials of degree k for ξ^μ, i.e.,

$$P_k^\mu := \{f \in \Pi_k\colon \langle f, p \rangle_\mu = 0, \ \forall p \in \Pi_{k-1}\}.$$

As expected, the eigenvalues of U_n are

$$\lambda_k(U_n) := \frac{n!}{(n-k)!} \frac{(n-1)!}{(n+k-1)!} = \lim_{\mu \to -1^+} \lambda_k(M_n^\mu), \quad k = 0, 1, \ldots, n.$$

This is easily seen for $k \geqslant 2$ since here P_k^μ converges (in the gap metric) to some P_k^* (see [W01]). Recall the gap between (finite-dimensional) subspaces P and Q of $C(T)$ is given by

$$\operatorname{gap}(P, Q) := \max\{\operatorname{dist}(P \cap B, Q), \operatorname{dist}(Q \cap B, P)\},$$
$$\operatorname{dist}(F, G) := \sup_{f \in F} \inf_{g \in G} \|f, g\|_{\infty, T},$$

where B is the unit ball in $C(T)$. Thus, for each $\mu > -1$ we can choose a basis $\{p_i^\mu\}$ of P_k^μ with $p_i^\mu \to p_i^*$ where $\{p_i^*\}$ is a basis for P_k^*, and so

$$M_n^\mu p_i^\mu = \lambda_k(M_n^\mu) p_i^\mu \quad \Rightarrow \quad U_n p_i^* = \lambda_k(U_n) p_i^*,$$

which implies P_k^* is the $\lambda_k(U_n)$-eigenspace (a dimension count shows it is all of it). For $k = 0, 1$ the limit eigenvalues are 1 (the rest are distinct). Here P_1^μ does not converge as $\mu \to -1^+$, though P_0^μ does and $P_0^\mu + P_1^\mu = \Pi_1$, which is easily seen by considering the functions

$$\xi_v - \frac{\mu_v + 1}{|\mu| + s + 1} \in P_1^\mu, \quad v \in V.$$

However, a simple calculation shows that Π_1 are the eigenfunctions of U_n for $\lambda = 1$, and so U_n is diagonalisable. The fact that linear polynomials are reproduced by U_n (as with B_n), was seen as desirable in [S94].

5. A differentiation formula for M_n^μ and U_n

In this final section, we give a formula for the derivative of $M_n^\mu f$ and $U_n f$ in terms of some $\hat{M}_{n-1}^\kappa$ applied to the derivative of f. Previously, see, e.g., [S94, Lemma 4.4] and [Di95, Property F], formulae for the derivative of $M_n^\mu f$ in terms of some operator applied to the derivative of f were given, but the operator was not identified.

The derivative of f in the direction $y \in \mathbb{R}^s$ is given by

$$D_y f := \lim_{t\to 0} \frac{f - f(\cdot + ty)}{t}.$$

Theorem 5.1 (Differentiation formula). *For $\mu \geqslant -1$ and $v, w \in V$, we have*

$$D_{v-w}(\hat{M}_n^\mu f) = \frac{n}{n+|\mu|+s+1} \hat{M}_{n-1}^{\mu+e_v+e_w}(D_{v-w} f), \quad \forall f \in C^1(T). \tag{5.1}$$

In particular,

$$D_{v-w}(U_n f) = \hat{M}_{n-1}^{e_v+e_w-1}(D_{v-w} f), \quad \forall f \in C^1(T). \tag{5.2}$$

Proof. In view of Theorem 3.2, it suffices to prove (5.2) for $\mu > -1$. Since

$$D_{v-w}(\xi^\alpha) = \alpha_v \xi^{\alpha-e_v} - \alpha_w \xi^{\alpha-e_w}, \tag{5.3}$$

we have

$$\begin{aligned} D_{v-w}(M_n^\mu f) &= \sum_{|\alpha|=n} \binom{n}{\alpha} \{\alpha_v \xi^{\alpha-e_v} - \alpha_w \xi^{\alpha-e_w}\} \frac{\langle f, \xi^\alpha \rangle_\mu}{\langle 1, \xi^\alpha \rangle_\mu} \\ &= \sum_{|\beta|=n-1} \binom{n}{\beta+e_v} (\beta_v+1) \xi^\beta \frac{\langle f, \xi^{\beta+e_v} \rangle_\mu}{\langle 1, \xi^{\beta+e_v} \rangle_\mu} \\ &\quad - \sum_{|\beta|=n-1} \binom{n}{\beta+e_w} (\beta_w+1) \xi^\beta \frac{\langle f, \xi^{\beta+e_w} \rangle_\mu}{\langle 1, \xi^{\beta+e_w} \rangle_\mu} \\ &= \sum_{|\beta|=n-1} \binom{n-1}{\beta} \left\{ \frac{n\langle f, \xi^{\beta+e_v} \rangle_\mu}{\langle 1, \xi^{\beta+e_v} \rangle_\mu} - \frac{n\langle f, \xi^{\beta+e_w} \rangle_\mu}{\langle 1, \xi^{\beta+e_w} \rangle_\mu} \right\}. \end{aligned}$$

By the beta integral (2.1),

$$\frac{n}{\langle 1, \xi^{\beta+e_v} \rangle_\mu} = \frac{n}{n+|\mu|+s+1} \frac{\beta_w + \mu_w + 1}{\langle 1, \xi^\beta \rangle_{\mu+e_v+e_w}},$$

and so we obtain

$$D_{v-w}(M_n^\mu f) = \frac{n}{n+|\mu|+s+1} \sum_{|\beta|=n-1} \binom{n-1}{\beta} \frac{c(f,\mu,\beta,v,w)}{\langle 1, \xi^\beta \rangle_{\mu+e_v+e_w}},$$

where

$$c(f,\mu,\beta,v,w) := \langle f, (\beta_w+\mu_w+1)\xi^{\beta+e_v}\rangle_\mu - \langle f,(\beta_v+\mu_v+1)\xi^{\beta+e_w}\rangle_\mu$$
$$= \int_V f\{(\beta_w+\mu_w+1)\xi^{\beta+\mu+e_v} - (\beta_v+\mu_v+1)\xi^{\beta+\mu+e_w}\}.$$

Using (5.3) and the integration by parts formula, we then have

$$c(f,\mu,\beta,v,w) = \int_V fD_{w-v}(\xi^{\beta+\mu+e_v+e_w})$$
$$= -\int_V D_{w-v}(f)\,\xi^{\beta+\mu+e_v+e_w} = \langle D_{v-w}f, \xi^{\beta+e_v+e_w}\rangle_\mu,$$

as required. □

In the univariate case $s=1$, $D_{v-w}=(v-w)D$, with D the univariate derivative, and $\mu+e_v+e_w=\mu+1$ (in case $v\neq w$), so the formula for kth derivatives, $k=1,\dots,n$, takes the simple form

$$D^k(U_nf) = M_{n-k}^{k-1}(D^kf), \quad \forall f\in C^k(T).$$

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